Rossmoyne Senior High School

Semester One Examination, 2019

Question/Answer booklet

MATHEMATICS METHODS YEAR 12 (ATMAM)

Section Two: Calculator-assumed

Circle your Teacher's Name: Alvaro Koulianos Bestall Luzuk Fraser-Jones Murray Kigodi Tanday

Student number: In figures

In words

Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	13	13	100	98	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answer to the specific question asked and to follow any instructions that are specified to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section Two: Calculator-assumed **65% (98 Marks)**

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 9 **(5 marks)**

Fuel flows into a storage tank that is initially empty at a rate of $\sqrt{4+3t}$ litres per minute, where t is the time in minutes and $0 \le t \le 100$.

(a) Determine how much fuel is in the tank after 20 minutes. (2 marks)

Solution
$V = \int_0^{20} \sqrt{4+3t}\, dt = 112\ \mathrm{L}$
Specific behaviours
✓ writes integral ✓ evaluates integral

(b) If the tank is completely full after 100 minutes, determine the time required for the tank to become one-quarter full. (3 marks)

Solution
$V = \int_0^{100} \sqrt{4+3t}\, dt = 1176.09\ \mathrm{L}$ $\int_0^{T} \sqrt{4+3t}\, dt = \frac{1176.09}{4}$ $\frac{2}{9}\left(\sqrt{4+3T}\right)^{\frac{3}{2}} - \frac{16}{9} = 294.02 \Rightarrow T = 39.0\ \text{minutes}$
Specific behaviours
✓ calculates total volume ✓ writes integral and equates to quarter volume ✓ evaluates time

Question 10 **(7 marks)**

X is a uniform discrete random variable where $x = 2, 3, 5, 7, 11, 13$.

(a) Determine

(i) $P(X \geq 5)$. (1 mark)

Solution
$P(X \geq 5) = \frac{4}{6} = \frac{2}{3}$
Specific behaviours
✓ correct value

(ii) $P(X < 12 \mid X \geq 3)$. (2 marks)

Solution
$P(X \geq 3) = \frac{5}{6}$ $P(X < 12 \vert X \geq 3) = \frac{4}{6} \div \frac{5}{6} = \frac{4}{5} = 0.8$
Specific behaviours
✓ $P(X \geq 3)$ ✓ correct probability

(b) Calculate the exact value of

(i) $\mathrm{E}(X)$. (2 marks)

Solution
$\mathrm{E}(X) = \frac{2+3+5+7+11+13}{6}$ $= \frac{41}{6} = 6.8\overline{3}$
Specific behaviours
✓ expression ✓ $\mathrm{E}(X)$ in exact form

(ii) $\mathrm{Var}(X)$. (2 marks)

Solution
$\sigma_X = \frac{\sqrt{581}}{6} \approx 4.017, \quad \sigma_X^2 = \frac{581}{36} \approx 16.14$
Specific behaviours
✓ standard deviation ✓ $\mathrm{Var}(X)$ in exact form

Question 11 **(8 marks)**

The potential difference, V volts, across the terminals of an electrical capacitor t seconds after it begins to discharge through a resistor can be modelled by the equation

$$V = V_0 e^{-kt}$$

V_0 is the initial potential difference and k is a constant that depends on the size of the capacitor and the resistor.

(a) If $V_0 = 22.6$ volts and $k = 0.018$, determine

(i) the potential difference across the capacitor 4 minutes after discharge began. (2 marks)

Solution
$V(240) = 0.30$ volts
Specific behaviours
✓ uses correct time ✓ calculates correct voltage

(ii) the time taken for the potential difference to drop from 17.5 to 12.5 volts. (3 marks)

Solution
When $V = 17.5, t = 14.2$ and when $V = 12.5, t = 32.9$. Hence takes $32.9 - 18.7 = 18.7\ s$ Alternative solution: $12.5 = 17.5e^{-0.018t}$, solve for t
Specific behaviours
✓ calculates first time ✓ calculates second time ✓ calculates difference, correct to at least 1 dp

(iii) the rate of change of V when the potential difference is 20 volts. (1 mark)

Solution
$\frac{dV}{dt} = -kV = -0.018 \times 20 = -0.36$ volts/sec
Specific behaviours
✓ calculates rate

(b) Another capacitor takes 66 seconds for its maximum potential difference to halve. It is instantly recharged to its maximum every 3 minutes, which is the time required for the potential difference to fall from its maximum to 1.8 volts. Determine the maximum potential difference for this capacitor. (2 marks)

Solution
$e^{-66k} = 0.5 \Rightarrow k = 0.0105$ $1.8 = V_0 e^{-0.0105 \times 180} \Rightarrow V_0 = 11.92$ volts
Specific behaviours
✓ determines k ✓ determines V_0 **NB:** Can solve with simultaneous equations and not show k .

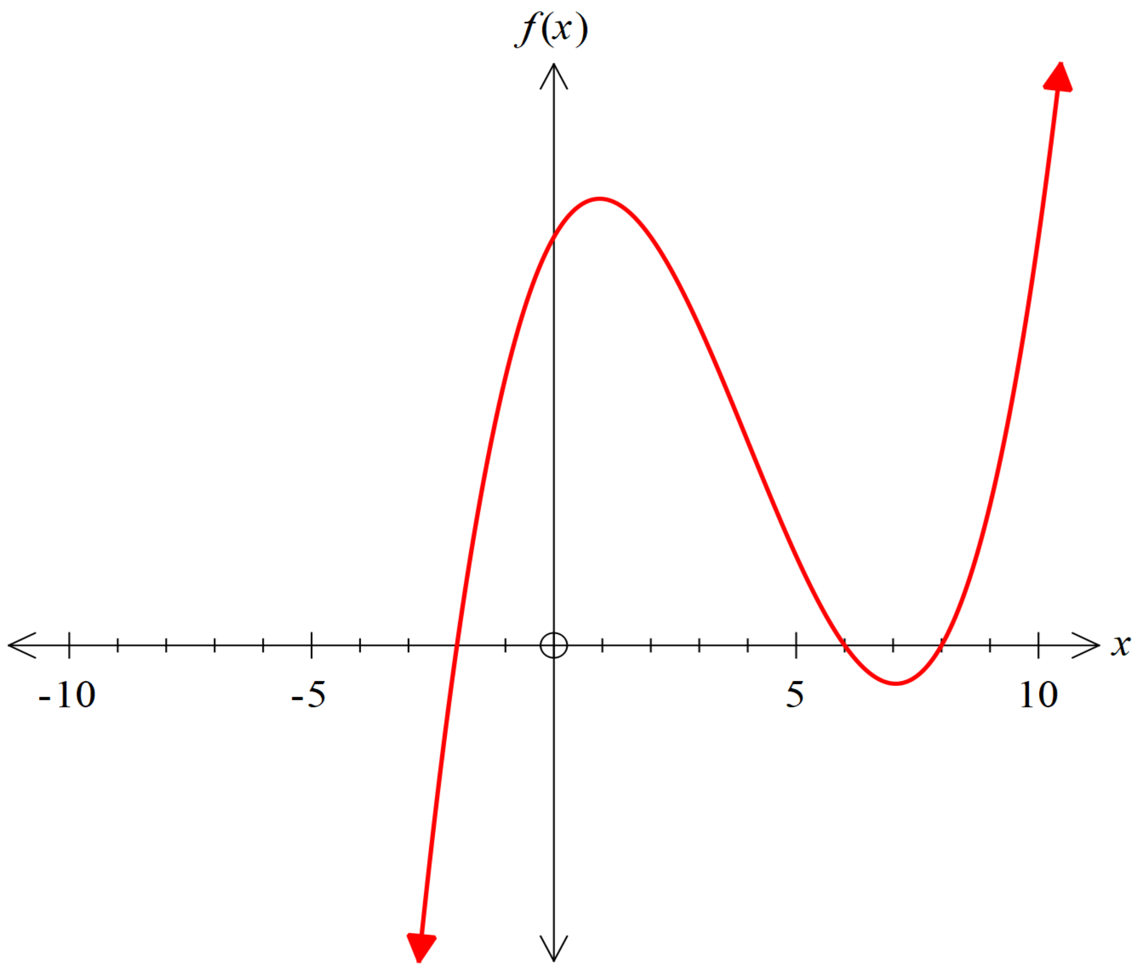
Question 12 **(6 marks)**

(a) Draw a graph that satisfies all the conditions listed below. Label the critical features clearly.

$f(-2) = f(6) = f(8) = 0$ (3 marks)

$f''(4) = 0$ and $f''(x) < 0$ for $x < 4$ only

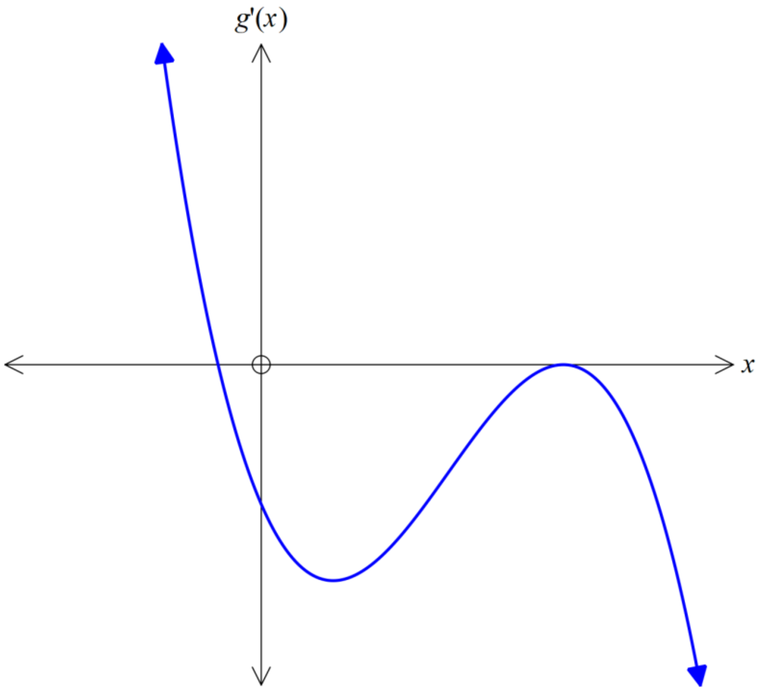
$f'(1) = f'(7) = 0$

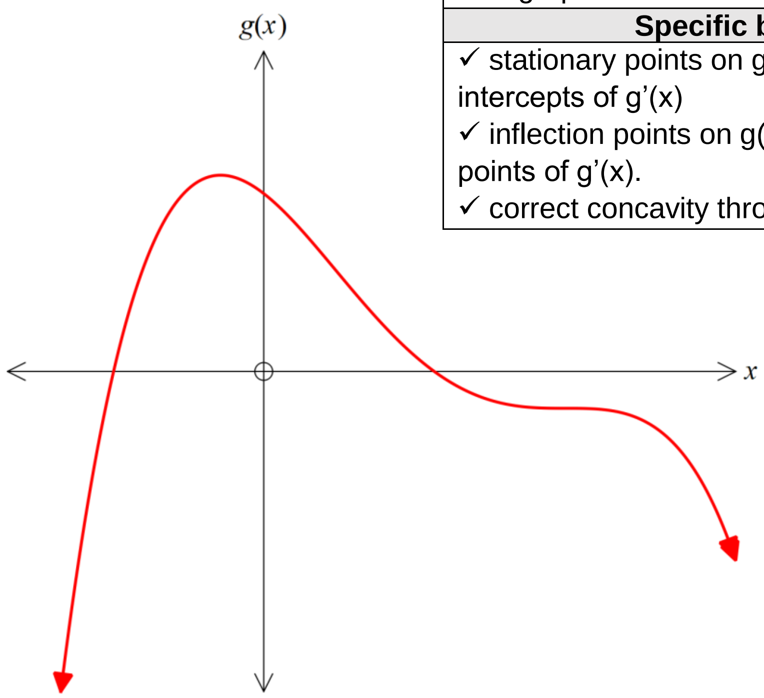


Solution
See graph
Specific behaviours
✓ correct x-coordinates of intercepts and positive y-coordinate ✓ correct x-coordinates of stationary points ✓ correct x-coordinate of inflection point and labelled as "POI"

(b) The graph of a gradient function is shown below. On the set of axes provided sketch a possible graph of its antiderivative.

(3 marks)





Solution
See graph
Specific behaviours
✓ stationary points on g(x) align with the x-intercepts of g'(x) ✓ inflection points on g(x) align with stationary points of g'(x). ✓ correct concavity throughout

Question 13 **(8 marks)**

A manufacturing process begins and the rate at which it produces gas after t minutes ($t \geq 0$) is modelled by

$$r(t) = 45(1 - e^{-0.4t})\ \mathrm{m}^3/\mathrm{minute}$$

(a) State the maximum rate that gas can be produced at. (1 mark)

Solution
$45\ \mathrm{m}^3/\mathrm{minute}$
Specific behaviours
✓ correct rate

(b) Calculate the rate that gas is being produced after 2 minutes. (1 mark)

Solution
$r(1) = 45(1 - e^{-0.8}) = 24.78\ \mathrm{m}^3/\mathrm{minute}$
Specific behaviours
✓ correct rate (exact or at least 1dp)

(c) Use the increments formula to determine the approximate change in r between 30 and 33 seconds after production began. (3 marks)

Solution
$\delta r \approx \frac{dr}{dt}\delta t \approx 18e^{-0.4t} \times \delta t$ $\approx \frac{18}{e^{0.2}} \times \frac{3}{60}$ $\approx \frac{9}{10e^{0.2}} \approx 0.7369\ \mathrm{m}^3/\mathrm{minute}$
Specific behaviours
✓ correct $r'(t)$ ✓ correct δt ✓ correct change

(d) Use the increments formula to determine the approximate volume of gas produced in the 5 seconds following $t = 2$. (3 marks)

Solution
$\delta V \approx \frac{dV}{dt}\delta t$ $\approx r(t) \times \delta t$ $\approx 24.78 \times \frac{5}{60}$ $\approx 2.065\ \mathrm{m}^3$
Specific behaviours
✓ correct use of increments formula   ✓ uses correct t and δt ✓ correct estimate (at least 2dp)

Question 14 **(7 marks)**

A small body has displacement $x = 0$ when $t = 8$ and moves along the x-axis so that its velocity after t seconds is given by

$$v(t) = 10\sin\left(\frac{\pi t}{24}\right)\ \text{cm/s}$$

(a) Determine an equation for $x(t)$, the displacement of the body after t seconds. (3 marks)

Solution
$$x = -\frac{10 \times 24}{\pi}\cos\left(\frac{\pi t}{24}\right) + c$$ $$t = 8 \Rightarrow 0 = \frac{-240}{\pi}\cos\left(\frac{\pi}{3}\right) + c \Rightarrow c = \frac{120}{\pi}$$ $$x = \frac{120}{\pi} - \frac{240}{\pi}\cos\left(\frac{\pi t}{24}\right)$$
Specific behaviours
✓ integrates v correctly ✓ attempts to find constant using substitution ✓ correct equation

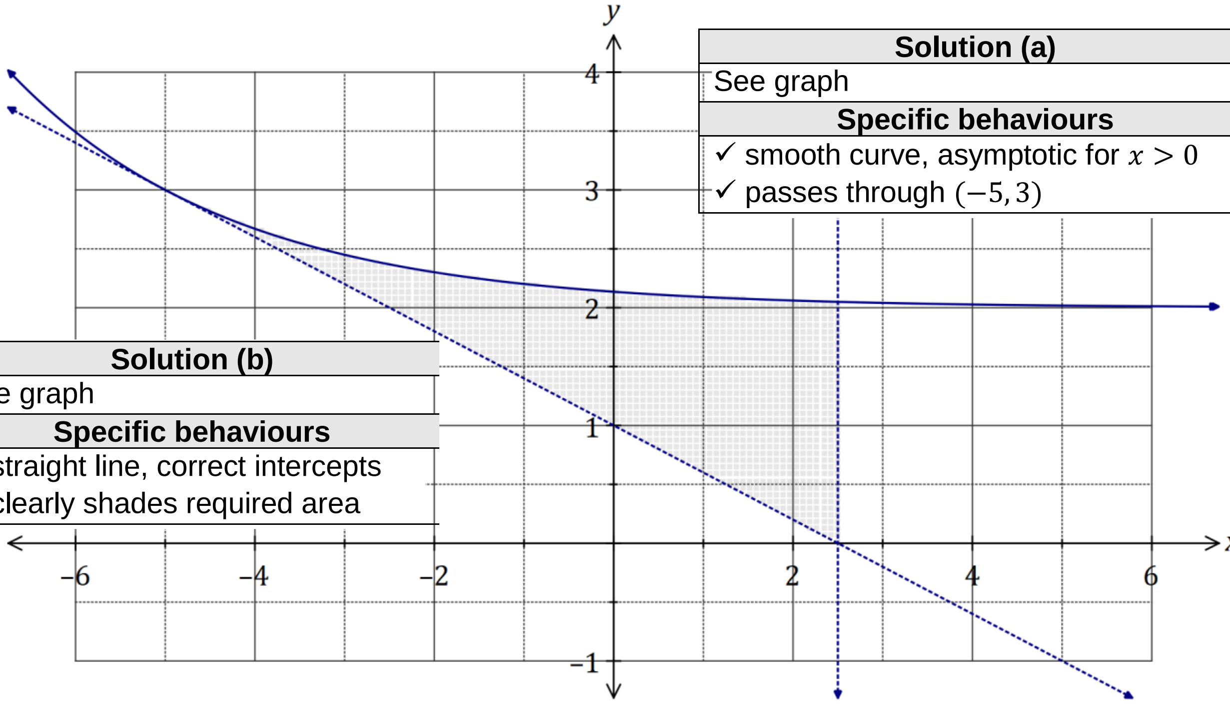
(b) Describe, with justification, how the speed of the body is changing when $t = 32$. (4 marks)

Solution
$$v(32) = 10\sin\left(\frac{4\pi}{3}\right) = -5\sqrt{3}$$ $$a = \frac{5\pi}{12}\cos\left(\frac{\pi t}{24}\right)$$ $$a(32) = \frac{5\pi}{12}\cos\left(\frac{4\pi}{3}\right) = -\frac{5\pi}{24}$$ Since the body has a negative velocity and a negative acceleration then its speed is increasing when $t = 32$.
Specific behaviours
✓ clearly shows v is negative ✓ expression for a ✓ clearly shows a is negative ✓ explains increasing speed using signs of v and a

Question 15 **(7 marks)**

Let $f(x) = 2 + e^{-0.4x-2}$.

(a) Sketch the graph of $y = f(x)$ on the axes below. (2 marks)



Solution (a)
See graph
Specific behaviours
✓ smooth curve, asymptotic for $x > 0$
✓ passes through $(-5, 3)$

Solution (b)
See graph
Specific behaviours
✓ straight line, correct intercepts
✓ clearly shades required area

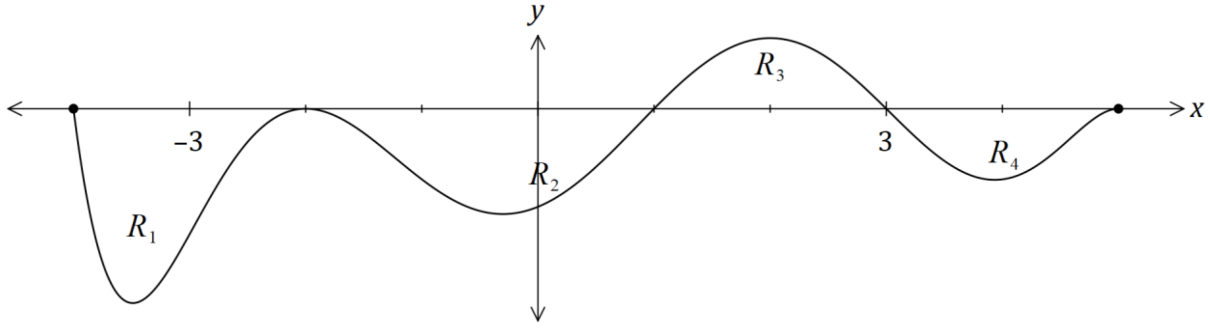
(b) The line $y = 1 - 0.4x$ is a tangent to the curve $y = f(x)$ at $x = -5$, and it intersects the x-axis at the point $(k, 0)$. Add the line to the graph above and shade the area enclosed by the line, the curve and $x = k$. (2 marks)

(c) Determine the area enclosed by the line, the curve and $x = k$. (3 marks)

Solution
$1 - 0.4k = 0 \Rightarrow k = 2.5$ $A = \int_{-5}^{2.5} (2 + e^{-0.4x-2}) - (1 - 0.4x)dx$ $= \dfrac{25}{4} - \dfrac{5e^{-3}}{2} \approx 6.126 \text{ sq units}$
Specific behaviours
✓ indicates value of k ✓ writes integral using difference of functions   ✓ evaluates integral

Question 16 **(7 marks)**

The graph of $y = f(x)$ is shown below for $-4 \leq x \leq 5$.



The area trapped between the x-axis and the curve for regions R_1, R_2, R_3 and R_4 are $35, 52, 28$ and 24 square units respectively.

(a) Determine the value of

(i) $\int_{-4}^{-2} f(x)\,dx$. (1 mark)

Solution
-35
Specific behaviours
✓ correct value

(ii) $\int_{-2}^{5} f(x)\,dx$. (2 marks)

Solution
$-52 + 28 - 24 = -48$
Specific behaviours
✓ shows sum of signed areas ✓ correct value

(iii) $\int_{1}^{5} (f(x) - 7)\,dx$. (2 marks)

Solution
$(28 - 24) - 4 \times 7 = 4 - 28 = -24$
Specific behaviours
✓ area of rectangle is 28 ✓ correct value

(iv) $\int_{-4}^{1} f(x)\,dx - \int_{1}^{5} f'(x)\,dx$. (2 marks)

Solution
$-35 - 52 - (0 - 0) = -87$
Specific behaviours
✓ shows second integral is zero ✓ correct value

Question 17 **(9 marks)**

Seeds were planted in rows of five and the number of seeds that germinated in each of the 120 rows are summarised below.

Number of germinating seeds	0	1	2	3	4	5
Number of rows	1	1	3	16	46	53

(a) Use the results in the table to determine

(i) the probability that no more than 4 seeds germinated in a randomly selected row. (1 mark)

Solution
$P(X \leq 4) = \dfrac{67}{120} \quad (= 0.558\overline{3})$
Specific behaviours
✓ correct probability

(ii) the mean number of seeds that germinated per row. (1 mark)

Solution
$\bar{x} = 4.2$
Specific behaviours
✓ correct mean

(b) Another row of five seeds is planted. Determine the probability that no more than 4 seeds germinate in **this** row. Assume the number that germinate per row is binomially distributed with the above mean. (2 marks)

Solution
$5p = 4.2 \Rightarrow p = \dfrac{4.2}{5} = 0.84$ $Y \sim B(5, 0.84)$ $P(X \leq 4) = 0.5818$
Specific behaviours
✓ calculates p ✓ correct probability

Suppose it is known that 66% of all seeds planted will germinate and that seeds are now planted in rows of 16.

(c) Assuming that seeds germinate independently of each other, determine

(i) the most likely number of seeds to germinate in a row. (1 mark)

Solution
11 seeds
Specific behaviours
✓ correct number **NB:** Accept 10

(ii) the probability that at least 9 seeds germinate in a randomly chosen row. (2 marks)

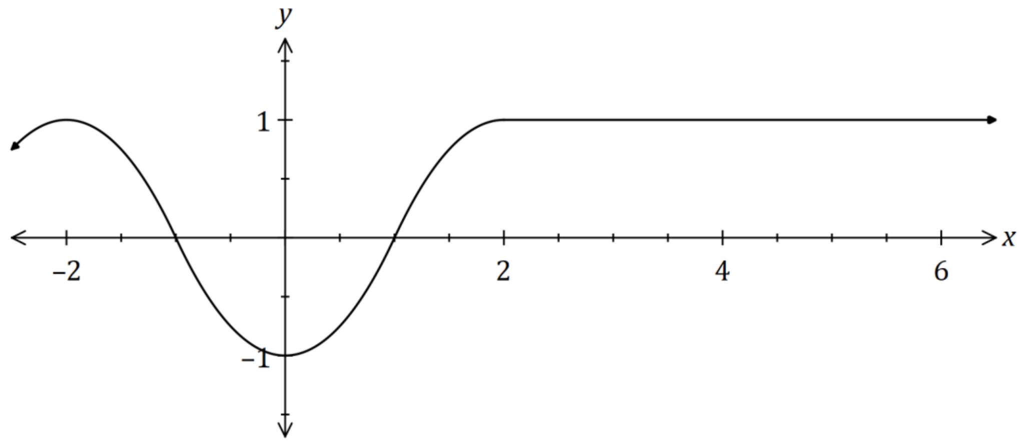
Solution
$W \sim B(16, 0.66)$ $P(W \geq 9) = 0.8609$
Specific behaviours
✓ states distribution ✓ correct probability

(iii) the probability that in eight randomly chosen rows, exactly six rows have at least 9 seeds germinating in them. (2 marks)

Solution
$V \sim B(8, 0.8609)$ $P(V = 6) = 0.2206$
Specific behaviours
✓ states distribution ✓ correct probability

Question 18 **(9 marks)**

The graph of $y = f(x)$ is shown below.



Let $A(x)$ be defined by the integral $A(x) = \int_{-2}^{x} f(t)\,dt$ for $x \geq -2$.

(a) Use the graph of $y = f(x)$ to identify all the turning points of the graph of $y = A(x)$, stating the x-coordinate and nature of each point. (2 marks)

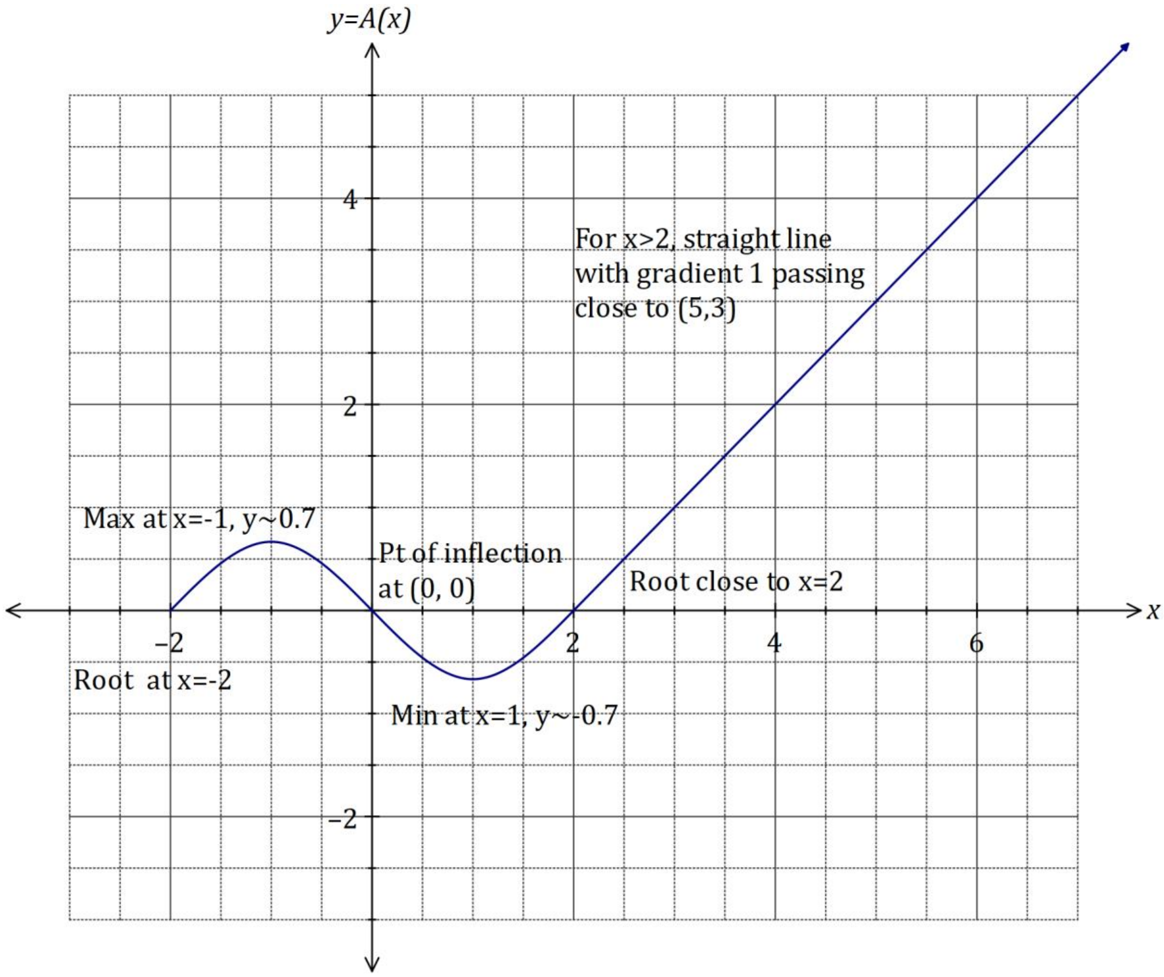
Solution
At $x = -1$ there is a maximum At $x = 1$ there is a minimum
Specific behaviours
✓ location of maximum ✓ location of minimum

It is also known that the $A(2) = 0$.

(b) Using the graph of $y = f(x)$ or otherwise, explain why $A(5) = 3$. (2 marks)

Solution
$A(5) = A(2) + \int_2^5 f(x)dx$. From the graph $\int_2^5 f(x)dx = 1 \times 3 = 3$, and hence $A(5) = 0 + 3 = 3$.
Specific behaviours
✓ shows use of $A(2)$ and integral ✓ explanation, using area and $A(2)$

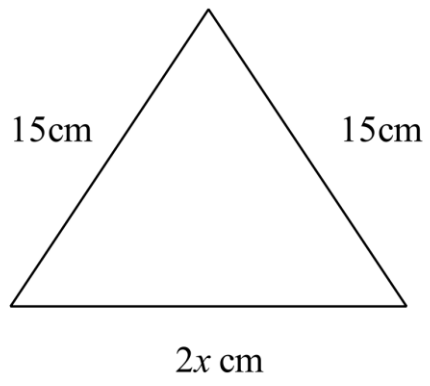
(c) Sketch the graph of $y = A(x)$ on the axes below, indicating and labelling the location of all key features. (5 marks)



Solution
See graph
Specific behaviours
✓ Labelled point of inflection at origin ✓ Labelled roots, as indicated ✓ Curve $-2 < x < 0$ with labelled maximum between 0.5 and 1 ✓ Curve $0 < x < 2$ with labelled minimum between -0.5 and -1 ✓ Straight line, as indicated **NB:** Deduct 1 mark only if TPs have y-values out of range.

Question 19 **(6 marks)**

A triangle has dimensions as shown below:



(a) Show that $A = x\sqrt{225 - x^2}$. (2 marks)

Solution
Let the perpendicular height be h $h = \sqrt{15^2 - x^2}$ $A = \frac{1}{2}2xh$ $= x\sqrt{225 - x^2}$
Specific behaviours
✓ finds h in terms of x ✓ expresses area in terms of x

(a) Use calculus methods to find the maximum area of the triangle. (4 marks)

Solution
$A' = \sqrt{15^2 - x^2} + x\frac{1}{2\sqrt{15^2 - x^2}}(-2x)$ $= \sqrt{15^2 - x^2} - x^2\frac{1}{\sqrt{15^2 - x^2}}$ Let $A' = 0$ and solve for x $x = \frac{15\sqrt{2}}{2}$ ≈ 10.607 $A'(10) \approx 2.2$ $A'(11) \approx -1.7$ Therefore SP is max $A\left(\frac{15\sqrt{2}}{2}\right) = 112.5cm^2$
Specific behaviours
✓ finds $A'(x)$ ✓ solves for $A'(x) = 0$ ✓ checks that it is a maximum (second derivative or sign test) ✓ determines maximum area with units

Question 20 **(7 marks)**

(a) Given that $f(t) = \sin\left(3t + \frac{\pi}{3}\right)$ and $F(x) = \int_0^x f(t)\,dt$, determine the exact value of

(i) $F\left(\frac{\pi}{2}\right)$. (1 mark)

Solution
$F\left(\frac{\pi}{2}\right) = \frac{1-\sqrt{3}}{6}$
Specific behaviours
✓ correct value (CAS)

(ii) $F'\left(\frac{\pi}{2}\right)$. (2 marks)

Solution
$F'(x) = f(x)$ $f\left(\frac{\pi}{2}\right) = -\frac{1}{2}$
Specific behaviours
✓ recognises $F'(x) = f(x)$ ✓ correct value

(b) Given that $G(x) = \int_1^x g(t)\,dt$, $\frac{d^2G}{dx^2} = 4 + 3\sqrt{x}$ and $G(4) = 56$, determine $g(t)$. (4 marks)

Solution
$G'(x) = g(x)$ $G''(x) = g'(x) = 4 + 3\sqrt{x}$ $g(x) = 4x + 2x^{1.5} + c$ $G(4) = \int_1^4 4t + 2t^{1.5} + c\,dt$ $= \frac{274}{5} + 3c$ $\frac{274}{5} + 3c = 56 \Rightarrow c = \frac{2}{5}$ $g(t) = 4t + 2t^{\frac{3}{2}} + \frac{2}{5}$
Specific behaviours
✓ shows $G'(x) = g(x)$ ✓ uses $G''(x) = g'(x)$ to obtain $g(x)$ with constant c ✓ integrates again to obtain $G(4)$ ✓ evaluates constant c and writes expression for $g(t)$

Question 21 **(12 marks)**

The random variable X is the number of goals scored by a team in a soccer match, where

$$P(X = x) = \frac{2.2^x e^{-2.2}}{x!} \quad \text{for } x = 0, 1, 2, 3, \ldots \text{ to infinity}$$

(a) Show that the probability that the team scores at least one goal in a match is $P(X \geq 1) = 0.8892$. (2 marks)

Solution
$P(X = 0) = 0.1108$ $P(X > 0) = 1 - 0.1108 = 0.8892$
Specific behaviours
✓ $P(X = 0)$ ✓ correct probability

The random variable Y is the bonus each player is paid after a match, depending on the number of goals the team scored. For four or more goals \$500 is paid, for two or three goals \$250 is paid and for one goal \$100 is paid. No bonus is paid if no goals are scored.

(b) Complete the probability distribution table for Y. (3 marks)

Goals scored	$x = 0$	$x = 1$	$2 \leq x \leq 3$	$x \geq 4$
y (\$)	0	**100**	**250**	500
$P(Y = y)$	**0.1108**	**0.2438**	**0.4648**	0.1806

Solution
$P(Y = 100) = P(X = 1) = 0.2438$ $P(Y = 250) = 1 - 0.1108 - 0.2438 - 0.1806 = 0.4648$ (accept 0.4647)
Specific behaviours
✓ missing y values ✓ $P(Y = 0)$ and $P(Y = 100)$ ✓ $P(Y = 250)$

(c) Calculate

(i) the mean bonus paid per match. (2 marks)

Solution
$\bar{Y} = 0 + 24.38 + 116.20 + 90.30$ $= \$230.88$
Specific behaviours
✓ expression ✓ mean value that rounds to \$230.90

(ii) the standard deviation of the bonus paid per match. (2 marks)

Solution
$\sigma_Y^2 = 23332.4$ $\sigma_Y = \$152.75$
Specific behaviours
✓ variance ✓ standard deviation Use of CAS expected

(d) The owner of the team plans to increase the current bonuses by $\$50$ next season (so that the players will get a bonus of $\$50$ even when no goals are scored) and then further raise them by 12% the following season. Determine the mean and standard deviation of the bonus paid per match after both changes are implemented. (3 marks)

Solution
$Z = (Y + 50) \times 1.12$ $E(Z) = (230.88 + 50) \times 1.12 = \$314.59$ $\sigma_Z = 152.75 \times 1.12 = \$171.08$
Specific behaviours
✓ correct multiplier ✓ new mean ✓ new standard deviation

Supplementary page

Question number: _________